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Multilinear commutators of Calderón–Zygmund operators on Hardy-type spaces with non-doubling measures [☆]

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Abstract

Under the assumption that μ is a non-negative Radon measure on \mathbb{R}^d which only satisfies some growth condition, the authors obtain the boundedness in some Hardy-type spaces of multilinear commutators generated by Calderón–Zygmund operators or fractional integrals with $RBMO(\mu)$ functions, where the Hardy-type spaces are some appropriate subspaces, associated to the considered $RBMO(\mu)$ functions, of the Hardy space $H^1(\mu)$ of Tolsa.

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1. Introduction

We will work on the d -dimensional Euclidean space \mathbb{R}^d with a non-negative Radon measure μ which only satisfies the following growth condition: there exists a constant $C_0 > 0$ such that

$$\mu(B(x, r)) \leq C_0 r^n$$

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for all $x \in \mathbb{R}^d$ and $r > 0$, where $B(x, r) = \{y \in \mathbb{R}^d: |y - x| < r\}$, n is a fixed number and $0 < n \leq d$. Since the measure μ is not necessary to satisfy the doubling condition that there exists a constant $C > 0$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all $x \in \text{supp}(\mu)$ and $r > 0$, we call such \mathbb{R}^d a non-homogeneous space. It is well known that the doubling condition is a key assumption in the analysis on spaces of homogeneous type. However, in recent years, a lot of papers focus on the study of function spaces and the boundedness of Calderón–Zygmund operators in non-homogeneous spaces and indicate that many classical results still hold in non-homogeneous spaces; see [4–7, 10–13] and their references. The analysis on non-homogeneous spaces was proved to play a striking role in solving the long open Painlevé’s problem by Tolosa in [14]; see also [15] for more background.

The main purpose of this paper is to establish the boundedness in some Hardy-type spaces of multilinear commutators generated by Calderón–Zygmund operators or fractional integrals with $RBMO(\mu)$ functions. Before stating our results, we first recall some necessary notation and definitions.

Let K be a function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, y): x = y\}$ satisfying that for $x \neq y$,

$$|K(x, y)| \leq C|x - y|^{-n}, \quad (1.1)$$

and for $|x - y| \geq 2|x - x'|$,

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \frac{|x - x'|^\delta}{|x - y|^{n+\delta}}, \quad (1.2)$$

where $\delta \in (0, 1]$ and $C > 0$ is a constant. The Calderón–Zygmund operator associated to the above kernel K and the measure μ is formally defined by

$$T(f)(x) = \int_{\mathbb{R}^d} K(x, y)f(y) d\mu(y). \quad (1.3)$$

This integral may be not convergent for many functions. Thus we consider the truncated operators T_ϵ for $\epsilon > 0$ defined by

$$T_\epsilon(f)(x) = \int_{|x-y|>\epsilon} K(x, y)f(y) d\mu(y).$$

We say that T is bounded on $L^2(\mu)$ if the operators T_ϵ are bounded on $L^2(\mu)$ uniformly on $\epsilon > 0$.

By a cube $Q \subset \mathbb{R}^d$ we mean a closed cube whose sides parallel to the axes and we denote its side length by $l(Q)$. Let α and β_d be positive constants such that $\alpha > 1$ and $\beta_d > \alpha^n$. For a cube Q , we say that Q is (α, β) -doubling if $\mu(\alpha Q) \leq \beta\mu(Q)$, where αQ denotes the cube concentric with Q and having side length $\alpha l(Q)$. In what follows, for definiteness, if α and β are not specified, by a doubling cube we mean a $(2, 2^{d+1})$ -doubling cube. Especially, for any given cube Q , we denote by \tilde{Q} the smallest doubling cube in the family $\{2^k Q\}_{k \geq 0}$. For two cubes $Q_1 \subset Q_2$, set

$$K_{Q_1, Q_2} = 1 + \sum_{k=1}^{N_{Q_1, Q_2}} \frac{\mu(2^k Q_1)}{l(2^k Q_1)^n},$$

where N_{Q_1, Q_2} is the first positive integer k such that $l(2^k Q_1) \geq l(Q_2)$; see [10] for some basic properties of K_{Q_1, Q_2} .

Definition 1.1. Let $\rho > 1$ be some fixed constant. We say that a function $b \in L^1_{\text{loc}}(\mu)$ belongs to the space $RBMO(\mu)$ if there is a constant $B > 0$ such that

$$\sup_Q \frac{1}{\mu(\rho Q)} \int_Q |b(x) - m_{\tilde{Q}}(b)| d\mu(x) \leq B < \infty, \quad (1.4)$$

and if $Q_1 \subset Q_2$ are doubling cubes,

$$|m_{Q_1}(b) - m_{Q_2}(b)| \leq BK_{Q_1, Q_2}, \quad (1.5)$$

where the supremum is taken over all cubes centered at some point of $\text{supp } \mu$ and $m_{\tilde{Q}}(b)$ is the mean value of b on \tilde{Q} , namely,

$$m_{\tilde{Q}}(b) = \frac{1}{\mu(\tilde{Q})} \int_{\tilde{Q}} b(x) d\mu(x).$$

The minimal constant B in (1.4) and (1.5) is defined to be the $RBMO(\mu)$ norm of b and is denoted by $\|b\|_*$.

The space $RBMO(\mu)$ was introduced by Tolsa [11] and he proved there that the definition of $RBMO(\mu)$ is independent of the choices of numbers ρ . In the sequel, we will choose $\rho = 2$.

Let T be the Calderón–Zygmund operator defined by (1.3). In what follows, we will always assume that T is bounded on $L^2(\mu)$. We now fix a \tilde{T} , which is a weak limit as $\epsilon \rightarrow 0$ of some subsequence of the uniformly $L^2(\mu)$ bounded operators T_ϵ on $\epsilon > 0$; see [11, p. 141]. It is easy to deduce that \tilde{T} is still bounded on $L^2(\mu)$; moreover, for $f \in L^2(\mu)$ whose support is not all of \mathbb{R}^d ,

$$\tilde{T}(f)(x) = \int_{\mathbb{R}^d} K(x, y) f(y) d\mu(y)$$

with the same K as in T , which satisfies (1.1) and (1.2). For such a \tilde{T} , $m \in \mathbb{N}$ and $b_i \in RBMO(\mu)$, $i = 1, 2, \dots, m$, we formally define the multilinear commutator $T_{\vec{b}}$ by

$$T_{\vec{b}}(f)(x) = [b_m, [b_{m-1}, \dots, [b_1, \tilde{T}] \cdots]](f)(x). \quad (1.6)$$

A such type of multilinear commutators when μ is the d -dimensional Lebesgue measure was first introduced by Pérez and Trujillo-González in [9]. When μ is a non-doubling measure, it was proved in [10] for $m = 1$ and in [3] for $m > 1$ that if T is bounded on $L^2(\mu)$ and $b_i \in RBMO(\mu)$ for $i = 1, \dots, m$, then $T_{\vec{b}}$ is bounded on $L^p(\mu)$ for $1 < p < \infty$. But $T_{\vec{b}}$ is not bounded from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ even when μ is the d -dimensional Lebesgue measure and $m = 1$; see [8]. In this paper, motivated by [8], we will first prove that for any $m \in \mathbb{N}$, $T_{\vec{b}}$ is bounded from some subspace of $H^1(\mu)$ associated with \vec{b} into $L^1(\mu)$, in analogy with the result established by Pérez in [8] with μ being the d -dimensional Lebesgue measure.

In the sequel, for $1 \leq i \leq m$, we denote by C_i^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(i)\}$ of $\{1, 2, \dots, m\}$ with i different elements. For any $\sigma \in C_i^m$, the complementary sequence σ' is given by $\sigma' = \{1, 2, \dots, m\} \setminus \sigma$. Let $\vec{b} = (b_1, b_2, \dots, b_m)$ be a finite family of locally integrable functions. For all $1 \leq i \leq m$ and $\sigma = \{\sigma(1), \dots, \sigma(i)\} \in C_i^m$, we define

$$\begin{aligned} [b(x) - b(y)]_\sigma &= [b_{\sigma(1)}(x) - b_{\sigma(1)}(y)] \cdots [b_{\sigma(i)}(x) - b_{\sigma(i)}(y)], \\ [b(x) - m_Q(b)]_\sigma &= [b_{\sigma(1)}(x) - m_Q(b_{\sigma(1)})] \cdots [b_{\sigma(i)}(x) - m_Q(b_{\sigma(i)})], \end{aligned}$$

and

$$[m_R(b) - m_Q(b)]_\sigma = [m_R(b_{\sigma(1)}) - m_Q(b_{\sigma(1)})] \cdots [m_R(b_{\sigma(i)}) - m_Q(b_{\sigma(i)})],$$

where Q and R are cubes in \mathbb{R}^d and $x, y \in \mathbb{R}^d$. With this notation, we write

$$\|\vec{b}_\sigma\|_* = \|b_{\sigma(1)}\|_* \cdots \|b_{\sigma(i)}\|_*.$$

If $\sigma = \{1, \dots, m\}$, we simply write

$$\|\vec{b}\|_* = \|b_1\|_* \cdots \|b_m\|_*.$$

Definition 1.2. Let $\rho > 1$, $1 < p \leq \infty$ and $\gamma, \tau \in \mathbb{N}$. Suppose $b_i \in RBMO(\mu)$ for $i = 1, 2, \dots, \tau$. A function $h \in L^1_{\text{loc}}(\mu)$ is called a $(\vec{b}, p, \tau, \gamma)$ -atomic block if

- (a) there exists some cube R such that $\text{supp}(h) \subset R$;
- (b) $\int_{\mathbb{R}^d} h(y) d\mu(y) = 0$;
- (c) $\int_{\mathbb{R}^d} h(y) b_\sigma(y) d\mu(y) = 0$ for all $1 \leq i \leq \tau$ and $\sigma \in C_i^\tau$;
- (d) for $j = 1, 2$, there are functions a_j supported on cube $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $h = \lambda_1 a_1 + \lambda_2 a_2$, and

$$\|a_j\|_{L^p(\mu)} \leq [\mu(\rho Q_j)]^{1/p-1} K_{Q_j, R}^{-\gamma}.$$

Then we denote

$$|h|_{H_{\vec{b}, \tau, \gamma}^{1, p}(\mu)} = |\lambda_1| + |\lambda_2|.$$

We say that $f \in H_{\vec{b}, \tau, \gamma}^{1, p}(\mu)$ if there are $(\vec{b}, p, \tau, \gamma)$ -atomic blocks $\{h_k\}_{k \in \mathbb{N}}$ such that

$$f = \sum_{k=1}^{\infty} h_k$$

with $\sum_{k=1}^{\infty} |h_k|_{H_{\vec{b}, \tau, \gamma}^{1, p}(\mu)} < \infty$. The $H_{\vec{b}, \tau, \gamma}^{1, p}(\mu)$ norm of f is defined by

$$\|f\|_{H_{\vec{b}, \tau, \gamma}^{1, p}(\mu)} = \inf \left\{ \sum_k |b_k|_{H_{\vec{b}, \tau, \gamma}^{1, p}(\mu)} \right\},$$

where the infimum is taken over all the possible decompositions of f in $(\vec{b}, p, \tau, \gamma)$ -atomic blocks.

Remark 1.1. By an argument similar to that in [10], we easily see that the above definition is independent of the chosen constant $\rho > 1$. If $\tau = 0$, the space $H_{\vec{b}, \tau, \gamma}^{1, p}(\mu)$ is just the atomic Hardy space introduced by Tolsa in [10] when $\gamma = 1$ and when $\gamma > 1$ by the authors in [2], which was proved in [2, 10] to be the Hardy space $H^1(\mu)$ of Tolsa in [13] with equivalent norms; see also Definition 1.4 below. However, it is still open if the spaces $H_{\vec{b}, \tau, \gamma}^{1, p}(\mu)$ are equivalent for any fixed $\tau \in \mathbb{N}$ and different $\gamma \in \mathbb{N}$ and $1 < p \leq \infty$. But we have the following obvious properties that for any $\tau \in \mathbb{N}$, $1 < p \leq \infty$ and $\gamma_1, \gamma_2 \in \mathbb{N}$ with $1 \leq \gamma_1 < \gamma_2$,

$$H_{\vec{b}, \tau, \gamma_2}^{1, p}(\mu) \subset H_{\vec{b}, \tau, \gamma_1}^{1, p}(\mu) \subset H^1(\mu),$$

and for any $\tau, \gamma \in \mathbb{N}$ and $1 < p_1 < p_2 \leq \infty$,

$$H_{b,\tau,\gamma}^{1,\infty}(\mu) \subset H_{b,\tau,\gamma}^{1,p_2}(\mu) \subset H_{b,\tau,\gamma}^{1,p_1}(\mu) \subset H^1(\mu).$$

Here is one of the main results of this paper.

Theorem 1.1. *Let $1 < p \leq \infty$, $m \in \mathbb{N}$ and $b_i \in RBMO(\mu)$ for $i = 1, 2, \dots, m$. Let T and $T_{\vec{b}}$ be as in (1.3) and (1.6), respectively. Suppose that T is bounded on $L^2(\mu)$. Then the multilinear commutator $T_{\vec{b}}$ is bounded from $H_{b,m,m+1}^{1,p}(\mu)$ into $L^1(\mu)$.*

Remark 1.2. Let us consider the multilinear commutator of the fractional integral operator, $I_{\alpha;\vec{b}}$, defined by

$$I_{\alpha;\vec{b}}(f)(x) = \int_{\mathbb{R}^d} \prod_{i=1}^m [b_i(x) - b_i(y)] \frac{f(y)}{|x - y|^{n-\alpha}} d\mu(y), \quad (1.7)$$

where $m \in \mathbb{N}$, $b_i \in RBMO(\mu)$ for $i = 1, 2, \dots, m$ and $0 < \alpha < n$. By an argument similar to the proof of Theorem 1.1, we can prove the following result for $I_{\alpha;\vec{b}}$ and we omit the details by similarity.

Theorem 1.2. *Let $0 < \alpha < n$, $1 < p \leq \infty$, $m \in \mathbb{N}$, $b_i \in RBMO(\mu)$ for $i = 1, 2, \dots, m$, and $I_{\alpha;\vec{b}}$ be as in (1.7). Then $I_{\alpha;\vec{b}}$ is bounded from $H_{b,m,m+1}^{1,p}(\mu)$ into $L^{n/(n-\alpha)}(\mu)$.*

In [1], the authors proved that \tilde{T} is bounded on the Hardy space $H^1(\mu)$ if $\tilde{T}^*(1) = 0$. Motivated by this, we now consider the boundedness of $T_{\vec{b}}$ from $H_{b,m,m+2}^{1,p}(\mu)$ into $H^1(\mu)$ with the assumption that $T_{\vec{b}}^*(1) = 0$. Here, by $T_{\vec{b}}^*(1) = 0$, we mean that for any bounded function h with compact support satisfying (b) and (c) of Definition 1.2,

$$\int_{\mathbb{R}^d} T_{\vec{b}}(h)(x) d\mu(x) = 0. \quad (1.8)$$

We point out that for a such function h , it is easy to see that $h \in H_{b,m,m+2}^{1,p}(\mu)$ and therefore, $T_{\vec{b}}(h) \in L^1(\mu)$ by Theorem 1.1. Also, if $T_{\vec{b}}(h) \in H^1(\mu)$, then $T_{\vec{b}}(h)$ should satisfy (1.8) by the definition of the Hardy space $H^1(\mu)$; see [10,13] or Definition 1.4 below. Thus, in some sense, condition (1.8) is also necessary. We remark that this result is new even when μ is the d -dimensional Lebesgue measure.

Definition 1.3. Given $f \in L_{\text{loc}}^1(\mu)$, we set

$$M_{\Phi} f(x) = \sup_{\varphi \sim x} \left| \int_{\mathbb{R}^d} f \varphi d\mu \right|,$$

where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d)$ and satisfies

- (i) $\|\varphi\|_{L^1(\mu)} \leq 1$,
- (ii) $0 \leq \varphi(y) \leq |y - x|^{-n}$ for all $y \in \mathbb{R}^d$, and
- (iii) $|\nabla \varphi(y)| \leq |y - x|^{-(n+1)}$ for all $y \in \mathbb{R}^d$, where $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_d)$.

Based on Theorem 1.2 of Tolsa in [13], we define the Hardy space $H^1(\mu)$ as follows.

Definition 1.4. The Hardy space $H^1(\mu)$ is the set of all functions $f \in L^1(\mu)$ satisfying that $\int_{\mathbb{R}^d} f d\mu = 0$ and $M_\Phi f \in L^1(\mu)$. Moreover, we define the norm of $f \in H^1(\mu)$ by

$$\|f\|_{H^1(\mu)} = \|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)}.$$

Another main result of this paper is as follows.

Theorem 1.3. Let $1 < p \leq \infty$, $m \in \mathbb{N}$ and $b_i \in RBMO(\mu)$ for $i = 1, 2, \dots, m$. Let T and $T_{\vec{b}}$ be as in (1.3) and (1.6), respectively. Suppose that T is bounded on $L^2(\mu)$ and $T_{\vec{b}}^*(1) = 0$ as in (1.8). Then $T_{\vec{b}}$ is bounded from $H_{\vec{b}, m, m+2}^{1,p}(\mu)$ into $H^1(\mu)$.

In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. For any index $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$.

2. Proof of Theorem 1.1

We begin with some necessary lemmas.

Lemma 2.1. Let $1 < p < \infty$, $m \in \mathbb{N}$ and $b_i \in RBMO(\mu)$ for $i = 1, 2, \dots, m$. Let T and $T_{\vec{b}}$ be as in (1.3) and (1.6), respectively. Suppose that T is bounded on $L^2(\mu)$. Then $T_{\vec{b}}$ is bounded on $L^p(\mu)$ with the norm no more than $C\|\vec{b}\|_*$, where $C > 0$ is a constant.

Lemma 2.1 was proved by Tolsa for the case $m = 1$ in [10] and by the authors for the cases $m > 1$ in [3]. The following Lemma 2.2 is a simple corollary of the John–Nirenberg inequality with non-doubling measure; see [10].

Lemma 2.2. Let $m \in \mathbb{N}$, $b_i \in RBMO(\mu)$ for $i = 1, 2, \dots, m$, $\rho > 1$ and $1 \leq p < \infty$. Then there exists a constant $C > 0$ such that for any cube Q ,

$$\left\{ \frac{1}{\mu(\rho Q)} \int_Q \prod_{i=1}^m |b_i(x) - m_{\vec{Q}}(b_i)|^p d\mu(x) \right\}^{1/p} \leq C \prod_{i=1}^m \|b_i\|_*.$$

Proof of Theorem 1.1. By a standard argument, it suffices to verify that for any $(\vec{b}, p, m, m+1)$ -atomic block h as in Definition 1.2 with $\rho = 4$, $T_{\vec{b}}(h)$ is in $L^1(\mu)$ with norm no more than $C|h|_{H_{\vec{b}, m, m+1}^{1,p}(\mu)}$, where $C > 0$ is a constant independent of h . Let all the notation be the same as in Definition 1.2. By our choices, a_j , $j = 1, 2$, now satisfies the following size condition:

$$\|a_j\|_{L^p(\mu)} \leq \mu(4Q_j)^{1/p-1} [K_{Q_j, R}]^{-(m+1)}. \quad (2.1)$$

Write

$$\int_{\mathbb{R}^d} |T_{\vec{b}}(h)(x)| d\mu(x) = \int_{2R} |T_{\vec{b}}(h)(x)| d\mu(x) + \int_{\mathbb{R}^d \setminus 2R} |T_{\vec{b}}(h)(x)| d\mu(x) = M + N.$$

To estimate M, we further decompose

$$M \leq \sum_{j=1}^2 |\lambda_j| \int_{2Q_j} |T_{\vec{b}}(a_j)(x)| d\mu(x) + \sum_{j=1}^2 |\lambda_j| \int_{2R_j \setminus 2Q_j} |T_{\vec{b}}(a_j)(x)| d\mu(x) = M_1 + M_2.$$

From the Hölder inequality, Lemma 2.1 and (2.1), it follows that

$$\begin{aligned} M_1 &\leq \sum_{j=1}^2 |\lambda_j| \|T_{\vec{b}}(a_j)\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \\ &\leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j|. \end{aligned}$$

Let $N_{j,1} = N_{2Q_j, 2R_j}$ for $j = 1, 2$. With the aid of the formula that for $x, y \in \mathbb{R}^d$,

$$\prod_{i=1}^m [b_i(x) - b_i(y)] = \sum_{i=0}^m \sum_{\sigma \in C_i^m} [b(x) - m_{\widetilde{Q}_j}(b)]_{\sigma} [m_{\widetilde{Q}_j}(b) - b(y)]_{\sigma'}, \quad (2.2)$$

where if $i = 0$, then $\sigma' = \{1, 2, \dots, m\}$ and $\sigma = \emptyset$, by (1.1), the Hölder inequality, Lemma 2.2 and (2.1), we easily obtain

$$\begin{aligned} M_2 &\leq \sum_{j=1}^2 |\lambda_j| \sum_{k=1}^{N_{j,1}} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \left| \int_{Q_j} \prod_{i=1}^m [b_i(x) - b_i(y)] K(x, y) a_j(y) d\mu(y) \right| d\mu(x) \\ &\leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=1}^{N_{j,1}} \int_{2^{k+1}Q_j \setminus 2^k Q_j} |[b(x) - m_{\widetilde{Q}_j}(b)]_{\sigma}| \\ &\quad \times \int_{Q_j} |[m_{\widetilde{Q}_j}(b) - b(y)]_{\sigma'}| \frac{|a_j(y)|}{|x - y|^n} d\mu(y) d\mu(x) \\ &\leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \left\{ \int_{Q_j} |a_j(y)|^p d\mu(y) \right\}^{1/p} \left\{ \int_{Q_j} |[m_{\widetilde{Q}_j}(b) - b(y)]_{\sigma'}|^{p'} d\mu(y) \right\}^{1/p'} \\ &\quad \times \left\{ \sum_{l=0}^i \sum_{\eta \in C_l^i} \sum_{k=1}^{N_{j,1}} \frac{1}{l(2^k Q_j)^n} \right. \\ &\quad \times \left. \int_{2^{k+1}Q_j} |[b(x) - m_{\widetilde{2^{k+1}Q_j}}(b)]_{\eta}] [m_{\widetilde{2^{k+1}Q_j}}(b) - m_{\widetilde{Q}_j}(b)]_{\eta'}| d\mu(x) \right\} \end{aligned}$$

$$\begin{aligned}
&\leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=1}^{N_{j,1}} \frac{\mu(2^{k+2}Q_j)}{l(2^kQ_j)^n} [K_{\widetilde{Q_j}, \widetilde{2^{k+1}Q_j}}]^i \\
&\leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} [K_{Q_j, R}]^{m+1} \\
&\leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j|,
\end{aligned}$$

where we have used the fact that for any $1 \leq k \leq N_{j,1}$, $K_{\widetilde{Q_j}, \widetilde{2^{k+1}Q_j}} \leq CK_{Q_j, R}$; see [10].

Now we turn to estimate N. Denote the center of R by x_R . The propositions (b) and (c) of Definition 1.2, (2.2), (1.2), the Hölder inequality, Lemma 2.2 and (2.1) lead to

$$\begin{aligned}
N &= \int_{\mathbb{R}^d \setminus 2R} \left| \int_R \prod_{i=1}^m [b_i(x) - b_i(y)] [K(x, y) - K(x, x_R)] h(y) d\mu(y) \right| d\mu(x) \\
&\leq C \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=1}^{\infty} \int_R |[b(y) - m_{\widetilde{R}}(b)]_{\sigma'}| |h(y)| \\
&\quad \times \int_{2^{k+1}R \setminus 2^kR} |[b(x) - m_{\widetilde{R}}(b)]_{\sigma}| \frac{|x - x_R|^\delta}{|x - y|^{n+\delta}} d\mu(x) d\mu(y) \\
&\leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \left\{ \sum_{l=0}^{m-i} \sum_{\eta \in C_l^{m-i}} \int_{Q_j} |[b(y) - m_{\widetilde{Q_j}}(b)]_{\eta}| [m_{\widetilde{Q_j}}(b) - m_{\widetilde{R}}(b)]_{\eta'} \right. \\
&\quad \times |[a_j(y)] d\mu(y) \Big\} \\
&\quad \times \left\{ \sum_{s=0}^i \sum_{\theta \in C_s^i} \sum_{k=1}^{\infty} \frac{l(R)^\delta}{l(2^kQ_j)^{n+\delta}} \right. \\
&\quad \times \left. \int_{2^{k+1}R} |[b(x) - m_{\widetilde{2^{k+1}R}}(b)]_{\theta}| [m_{\widetilde{2^{k+1}R}}(b) - m_{\widetilde{R}}(b)]_{\theta'} d\mu(x) \Big\} \\
&\leq C \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)}^p \sum_{i=0}^m \sum_{\sigma \in C_i^m} \|\vec{b}_\sigma\|_* \sum_{k=1}^{\infty} \frac{\mu(2^{k+2}R) l(R)^\delta}{l(2^kQ_j)^{n+\delta}} [K_{\widetilde{R}, \widetilde{2^{k+1}R}}]^i \\
&\quad \times \sum_{l=0}^{m-i} \sum_{\eta \in C_l^{m-i}} \left\{ \int_{Q_j} |[b(y) - m_{\widetilde{Q_j}}(b)]_{\eta}| [m_{\widetilde{Q_j}}(b) - m_{\widetilde{R}}(b)]_{\eta'} |^{p'} d\mu(y) \right\}^{1/p'} \\
&\leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} [K_{Q_j, R}]^m
\end{aligned}$$

$$\leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j|.$$

Combining the estimates for M and N, we have finished the proof of Theorem 1.1. \square

3. Proof of Theorem 1.3

In order to prove Theorem 1.3, we first recall the following basic fact in [1].

Lemma 3.1. *Let M_Φ be as in Definition 1.3 and $1 < p < \infty$. Then M_Φ is bounded on $L^p(\mu)$.*

Proof of Theorem 1.3. Just as in the proof of Theorem 1.1, we only need to verify that for any $(\vec{b}, p, m, m+2)$ -atomic block h as in Definition 1.2 with $\rho = 4$, $T_{\vec{b}}(h)$ is in $H^1(\mu)$ with norm no more than $C|h|_{H_{\vec{b}, m, m+2}^{1,p}(\mu)}$, where $C > 0$ is a constant independent of h . By our choices, for $j = 1, 2$, a_j satisfies the following size condition that

$$\|a_j\|_{L^p(\mu)} \leq \mu(4Q_j)^{1/p-1} [K_{Q_j, R}]^{-(m+2)}. \quad (3.1)$$

By the assumption that $T_{\vec{b}}^*(1) = 0$, Theorem 1.1 and Definition 1.4, we deduce that the proof of Theorem 1.3 can be reduced to proving that

$$\|M_\Phi[T_{\vec{b}}(h)]\|_{L^1(\mu)} \leq C \|\vec{b}\|_* |h|_{H_{\vec{b}, m, m+2}^{1,p}(\mu)}. \quad (3.2)$$

Write

$$\|M_\Phi[T_{\vec{b}}(h)]\|_{L^1(\mu)} = \int_{4R} M_\Phi[T_{\vec{b}}(h)](x) d\mu(x) + \int_{\mathbb{R}^d \setminus 4R} M_\Phi[T_{\vec{b}}(h)](x) d\mu(x) = \text{I} + \text{II}.$$

Noting that M_Φ is sublinear, we can control I by

$$\text{I} \leq \int_{4R} M_\Phi\{[T_{\vec{b}}(h)]\chi_{8R}\}(x) d\mu(x) + \int_{4R} M_\Phi\{[T_{\vec{b}}(h)]\chi_{\mathbb{R}^d \setminus 8R}\}(x) d\mu(x) = \text{I}_1 + \text{I}_2.$$

From the fact that for $j = 1, 2$, $Q_j \subset R$, it follows that for any $z \in Q_j$ and any $y \in 2^{k+1}R \setminus 2^kR$, $k \geq 3$, $|y - z| \geq l(2^{k-2}R)$. By this fact, (ii) of Definition 1.3, (2.2), (1.1), Lemma 2.2 and (3.1), we obtain

$$\begin{aligned} \text{I}_2 &\leq \int_{4R} \sup_{\varphi \sim x} \left[\int_{\mathbb{R}^d \setminus 8R} |T_{\vec{b}}(h)(y)| \varphi(y) d\mu(y) \right] d\mu(x) \\ &\leq \sum_{j=1}^2 |\lambda_j| \sum_{k=3}^{\infty} \int_{4R} \int_{2^{k+1}R \setminus 2^kR} \left| \int_{Q_j} \prod_{i=1}^{\infty} [b_i(y) - b_i(z)] K(y, z) a_j(z) d\mu(z) \right| \\ &\quad \times \frac{1}{|x - y|^n} d\mu(y) d\mu(x) \\ &\leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=3}^{\infty} \int_{4R} \int_{2^{k+1}R \setminus 2^kR} \frac{|[b(y) - m_{\vec{Q}_j}(b)]_\sigma|}{|x - y|^n} \end{aligned}$$

$$\begin{aligned}
& \times \int_{Q_j} \frac{|[m_{\widetilde{Q}_j}(b) - b(z)]_{\sigma'}| |a_j(z)|}{|y - z|^n} d\mu(z) d\mu(y) d\mu(x) \\
& \leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=3}^{\infty} \frac{\mu(4R)}{l(2^k R)^n} \frac{\|a_j\|_{L^p(\mu)}}{l(2^k R)^n} \left\{ \int_{Q_j} |[m_{\widetilde{Q}_j}(b) - b(z)]_{\sigma'}|^{p'} d\mu(z) \right\}^{1/p'} \\
& \quad \times \int_{2^{k+1}R} \left| [b(y) - m_{\widetilde{2^{k+1}R}}(b) + m_{\widetilde{2^{k+1}R}}(b) - m_{\widetilde{R}}(b) + m_{\widetilde{R}}(b) - m_{\widetilde{Q}_j}(b)]_{\sigma} \right| d\mu(y) \\
& \leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \\
& \quad \times \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=3}^{\infty} \frac{\mu(4R)\mu(2^{k+2}R)}{l(2^k R)^n l(2^k R)^n} \{ [K_{\widetilde{R}, \widetilde{2^{k+1}R}}]^i + [K_{\widetilde{Q}_j, \widetilde{R}}]^i \} \\
& \leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j|.
\end{aligned}$$

To estimate I_1 , we write

$$\begin{aligned}
I_1 & \leq \sum_{j=1}^2 |\lambda_j| \int_{4Q_j} M_{\Phi} \{ [T_{\widetilde{b}}(a_j)] \chi_{8R} \} (x) d\mu(x) \\
& \quad + \sum_{j=1}^2 |\lambda_j| \int_{4R \setminus 4Q_j} M_{\Phi} \{ [T_{\widetilde{b}}(a_j)] \chi_{2Q_j} \} (x) d\mu(x) \\
& \quad + \sum_{j=1}^2 |\lambda_j| \int_{4R \setminus 4Q_j} M_{\Phi} \{ [T_{\widetilde{b}}(a_j)] \chi_{8R \setminus 2Q_j} \} (x) d\mu(x) \\
& = I_{11} + I_{12} + I_{13}.
\end{aligned}$$

The Hölder inequality, Lemma 3.1, Lemma 2.1 and (3.1) lead to

$$\begin{aligned}
I_{11} & \leq \sum_{j=1}^2 |\lambda_j| \mu(4Q_j)^{1/p'} \|M_{\Phi} \{ [T_{\widetilde{b}}(a_j)] \chi_{8R} \}\|_{L^p(\mu)} \\
& \leq C \sum_{j=1}^2 |\lambda_j| \mu(4Q_j)^{1/p'} \|T_{\widetilde{b}}(a_j)\|_{L^p(\mu)} \\
& \leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(4Q_j)^{1/p'} \\
& \leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j|.
\end{aligned}$$

For $j = 1, 2$, denote $N_{Q_j, 4R}$ simply by $N_{j, 2}$. By (ii) of Definition 1.3, the Hölder inequality, Lemma 2.1 and (3.1), we have

$$\begin{aligned}
 I_{12} &\leq \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{N_{j,2}} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \sup_{\varphi \sim x} \left| \int_{2Q_j} T_{\vec{b}}(a_j)(y) \varphi(y) d\mu(y) \right| d\mu(x) \\
 &\leq C \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{N_{j,2}} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \frac{1}{l(2^k Q_j)^n} d\mu(x) \int_{2Q_j} |T_{\vec{b}}(a_j)(y)| d\mu(y) \\
 &\leq C \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{N_{j,2}} \frac{\mu(2^{k+1}Q_j)}{l(2^k Q_j)^n} \|T_{\vec{b}}(a_j)\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \\
 &\leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} K_{Q_j, R} \\
 &\leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j|.
 \end{aligned}$$

For I_{13} , we further decompose it into

$$\begin{aligned}
 I_{13} &\leq \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{N_{j,2}} \int_{2^{k+1}Q_j \setminus 2^k Q_j} M_{\Phi} [|T_{\vec{b}}(a_j)| \chi_{2^{k+2}Q_j \setminus 2^{k-1}Q_j}](x) d\mu(x) \\
 &\quad + \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{N_{j,2}} \int_{2^{k+1}Q_j \setminus 2^k Q_j} M_{\Phi} [|T_{\vec{b}}(a_j)| \chi_{\max\{2^{k+2}Q_j, 8R\} \setminus 2^{k+2}Q_j}](x) d\mu(x) \\
 &\quad + \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{N_{j,2}} \int_{2^{k+1}Q_j \setminus 2^k Q_j} M_{\Phi} [|T_{\vec{b}}(a_j)| \chi_{2^{k-1}Q_j \setminus 2Q_j}](x) d\mu(x) \\
 &= E + F + G.
 \end{aligned}$$

The Hölder inequality, Lemma 3.1, (2.2), (1.1), Lemma 2.2 and (3.1) tell us that

$$\begin{aligned}
 E &\leq \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{N_{j,2}} \mu(2^{k+1}Q_j)^{1/p'} \|M_{\Phi} [|T_{\vec{b}}(a_j)| \chi_{2^{k+2}Q_j \setminus 2^{k-1}Q_j}]\|_{L^p(\mu)} \\
 &\leq C \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{N_{j,2}} \mu(2^{k+1}Q_j)^{1/p'} \left\{ \int_{2^{k+2}Q_j \setminus 2^{k-1}Q_j} |T_{\vec{b}}(a_j)|^p d\mu(y) \right\}^{1/p} \\
 &\leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=2}^{N_{j,2}} \mu(2^{k+1}Q_j)^{1/p'} \left\{ \int_{2^{k+2}Q_j \setminus 2^{k-1}Q_j} |[b(y) - m_{\vec{Q}_j}(b)]_{\sigma}|^p \right.
 \end{aligned}$$

$$\begin{aligned}
& \times \left| \int_{Q_j} [m_{\widetilde{Q}_j}(b) - b(z)]_{\sigma'} K(y, z) a_j(z) d\mu(z) \right|^p d\mu(y) \Big\}^{1/p} \\
& \leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=2}^{N_{j,2}} \frac{\mu(2^{k+1}Q_j)^{1/p'}}{l(2^k Q_j)^n} \int_{Q_j} |[m_{\widetilde{Q}_j}(b) - b(z)]_{\sigma'}| |a_j(z)| d\mu(z) \\
& \quad \times \left\{ \int_{2^{k+2}Q_j} |[b(y) - m_{\widetilde{2^{k+2}Q_j}}(b) + m_{\widetilde{2^{k+2}Q_j}}(b) - m_{\widetilde{Q}_j}(b)]_{\sigma'}|^p d\mu(y) \right\}^{1/p} \\
& \leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \|\vec{b}_{\sigma}\|_* \sum_{k=2}^{N_{j,2}} \frac{\mu(2^{k+3}Q_j)}{l(2^k Q_j)^n} [K_{\widetilde{Q_j}, \widetilde{2^{k+2}Q_j}}]^{p'} \\
& \quad \times \left\{ \int_{Q_j} |a_j(z)|^p d\mu(z) \right\}^{1/p} \left\{ \int_{Q_j} |[m_{\widetilde{Q}_j}(b) - b(z)]_{\sigma'}|^{p'} d\mu(z) \right\}^{1/p'} \\
& \leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} [K_{Q_j, R}]^{m+1} \\
& \leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j|.
\end{aligned}$$

By (ii) of Definition 1.3, (2.2), (1.1), the Hölder inequality, Lemma 2.2 and (3.1), we easily see

$$\begin{aligned}
G & \leq \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{N_{j,2}} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \sup_{\varphi \sim x} \left[\int_{2^{k-1}Q_j \setminus 2Q_j} |T_{\vec{b}}(a_j)(y)| \varphi(y) d\mu(y) \right] d\mu(x) \\
& \leq \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{N_{j,2}} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \sum_{l=1}^{k-2} \int_{2^{l+1}Q_j \setminus 2^l Q_j} \frac{|T_{\vec{b}}(a_j)(y)|}{|y-x|^n} d\mu(y) d\mu(x) \\
& \leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=2}^{N_{j,2}} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \sum_{l=1}^{k-2} \int_{2^{l+1}Q_j \setminus 2^l Q_j} \frac{|[b(y) - m_{\widetilde{Q}_j}(b)]_{\sigma}|}{|y-x|^n} \\
& \quad \times \left| \int_{Q_j} [m_{\widetilde{Q}_j}(b) - b(z)]_{\sigma'} K(y, z) a_j(z) d\mu(z) \right| d\mu(y) d\mu(x) \\
& \leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=2}^{N_{j,2}} \frac{\mu(2^{k+1}Q)}{l(2^k Q_j)^n} \sum_{l=1}^{k-2} \frac{1}{l(2^l Q_j)^n} \int_{2^{l+1}Q_j} |[b(y) - m_{\widetilde{Q}_j}(b)]_{\sigma}| d\mu(y) \\
& \quad \times \left\{ \int_{Q_j} |a_j(z)|^p d\mu(z) \right\}^{1/p} \left\{ \int_{Q_j} |[m_{\widetilde{Q}_j}(b) - b(z)]_{\sigma'}|^{p'} d\mu(z) \right\}^{1/p'}
\end{aligned}$$

$$\begin{aligned}
&\leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=2}^{N_{j,2}} \frac{\mu(2^{k+1}Q)}{l(2^k Q_j)^n} \sum_{l=1}^{k-2} \frac{\mu(2^{l+2}Q_j)}{l(2^l Q_j)^n} [K_{\widetilde{Q_j}, 2^{l+1}Q_j}]^i \\
&\leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} [K_{Q_j, R}]^{m+2} \\
&\leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j|.
\end{aligned}$$

An argument similar to the estimate for G can also give us a desired estimate for F. The estimates for E, F and G lead us a desired estimate for I_{13} . Combining the estimates for I_{11} , I_{12} , I_{13} and I_2 yield

$$I = \int_{4R} M_\Phi [T_{\vec{b}}(h)](x) d\mu(x) \leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j| = C \|\vec{b}\|_* |h|_{H_{b,m,m+2}^{1,p}(\mu)}. \quad (3.3)$$

Now we turn to the estimate for II. Invoking that $T_{\vec{b}}^*(1) = 0$, we obtain

$$\begin{aligned}
II &= \int_{\mathbb{R}^d \setminus 4R} \sup_{\varphi \sim x} \left| \int_{\mathbb{R}^d} T_{\vec{b}}(h)(y) [\varphi(y) - \varphi(x_R)] d\mu(y) \right| d\mu(x) \\
&\leq \int_{\mathbb{R}^d \setminus 4R} \sup_{\varphi \sim x} \left| \int_{2R} T_{\vec{b}}(h)(y) [\varphi(y) - \varphi(x_R)] d\mu(y) \right| d\mu(x) \\
&\quad + \int_{\mathbb{R}^d \setminus 4R} \sup_{\varphi \sim x} \left| \int_{\mathbb{R}^d \setminus 2R} T_{\vec{b}}(h)(y) [\varphi(y) - \varphi(x_R)] d\mu(y) \right| d\mu(x) \\
&= II_1 + II_2.
\end{aligned}$$

Note that for any $z \in 2R$, $x \in 2^{k+1}R \setminus 2^kR$, $k \geq 2$, $|x - z| \geq l(2^{k-2}R)$. This together with (iii) of Definition 1.3 and the mean value theorem leads to

$$|\varphi(y) - \varphi(x_R)| \leq C \frac{l(R)}{l(2^{k-2}R)^{n+1}} \quad (3.4)$$

for $y \in 2R$. By (3.4), (2.2), (1.1), the Hölder inequality, Lemmas 2.1 and 2.2, and (3.1), we have

$$\begin{aligned}
II_1 &\leq \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^kR} \sup_{\varphi \sim x} \left[\int_{2R \setminus 2Q_j} |T_{\vec{b}}(a_j)(y)| |\varphi(y) - \varphi(x_R)| d\mu(y) \right] d\mu(x) \\
&\quad + \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^kR} \sup_{\varphi \sim x} \left[\int_{2Q_j} |T_{\vec{b}}(a_j)(y)| |\varphi(y) - \varphi(x_R)| d\mu(y) \right] d\mu(x) \\
&\leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^kR} \frac{l(R)}{l(2^kR)^{n+1}} \sum_{l=1}^{N_{j,2}-1} \int_{2^{l+1}Q_j \setminus 2^lQ_j} |[b(y) - m_{\widetilde{Q_j}}(b)]_\sigma|
\end{aligned}$$

$$\begin{aligned}
& \times \int_{Q_j} \left| [m_{\widetilde{Q}_j}(b) - b(z)]_{\sigma'} \right| \frac{|a_j(z)|}{|y-z|^n} d\mu(z) d\mu(y) d\mu(x) \\
& + C \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^k R} \frac{l(R)}{l(2^k R)^{n+1}} \| [T_{\widetilde{b}}(a_j)] \chi_{2Q_j} \|_{L^1(\mu)} d\mu(x) \\
& \leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=2}^{\infty} \frac{\mu(2^{k+1}R)l(R)}{l(2^k R)^{n+1}} \\
& \quad \times \sum_{l=1}^{N_{j,2}-1} \frac{1}{l(2^l Q_j)^n} \int_{2^{l+1}Q_j} |[b(y) - m_{\widetilde{Q}_j}(b)]_{\sigma}| d\mu(y) \\
& \quad \times \int_{Q_j} |a_j(z)| |[m_{\widetilde{Q}_j}(b) - b(z)]_{\sigma'}| d\mu(z) \\
& \quad + C \sum_{j=1}^2 |\lambda_j| \sum_{k=2}^{\infty} \frac{\mu(2^{k+1}R)l(R)}{l(2^k R)^{n+1}} \| [T_{\widetilde{b}}(a_j)] \chi_{2Q_j} \|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \\
& \leq C \|\widetilde{b}\|_* \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p} \sum_{k=2}^{\infty} \frac{\mu(2^{k+1}R)l(R)}{l(2^k R)^{n+1}} \\
& \quad \times \sum_{l=1}^{N_{j,2}-1} \frac{\mu(2^{l+2}Q_j)}{l(2^l Q_j)^n} [K_{\widetilde{Q}_j, 2^{l+1}Q_j}]^m \\
& \quad + C \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \\
& \leq C \|\widetilde{b}\|_* \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} [K_{Q_j, R}]^{m+1} \\
& \leq C \|\widetilde{b}\|_* \sum_{j=1}^2 |\lambda_j|.
\end{aligned}$$

We further estimate Π_2 by

$$\begin{aligned}
\Pi_2 &= \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^k R} \sup_{\varphi \sim x} \left| \int_{\mathbb{R}^d \setminus 2R} T_{\widetilde{b}}(h)(y) [\varphi(y) - \varphi(x_R)] d\mu(y) \right| d\mu(x) \\
&\leq \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^k R} M_{\Phi} [|T_{\widetilde{b}}(h)| \chi_{2^{k+2}R \setminus 2^{k-1}R}](x) d\mu(x) \\
&\quad + \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^k R} \sup_{\varphi \sim x} \left[\int_{2^{k+2}R \setminus 2^{k-1}R} |T_{\widetilde{b}}h(y)| \varphi(x_R) d\mu(y) \right] d\mu(x)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^k R} \sup_{\varphi \sim x} \left[\int_{\mathbb{R}^d \setminus 2^{k+2}R} |T_{\tilde{b}}(h)(y)| \{ \varphi(y) + \varphi(x_R) \} d\mu(y) \right] d\mu(x) \\
& + \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^k R} \sup_{\varphi \sim x} \left[\int_{2^{k-1}R \setminus 2R} |T_{\tilde{b}}(h)(y)| \{ \varphi(y) + \varphi(x_R) \} d\mu(y) \right] d\mu(x) \\
& = \Pi_{21} + \Pi_{22} + \Pi_{23} + \Pi_{24}.
\end{aligned}$$

From the Hölder inequality, Lemma 3.1, (b) and (c) of Definition 1.2, (2.2), (1.2), Lemma 2.2 and (3.1), we can deduce that

$$\begin{aligned}
\Pi_{21} & \leq \sum_{k=2}^{\infty} \mu(2^{k+1}R)^{1/p'} \|M_{\Phi}[|T_{\tilde{b}}(h)|\chi_{2^{k+2}R \setminus 2^{k-1}R}]\|_{L^p(\mu)} \\
& \leq C \sum_{k=2}^{\infty} \mu(2^{k+1}R)^{1/p'} \left\{ \int_{2^{k+2}R \setminus 2^{k-1}R} \left| \int_R \prod_{i=1}^m [b_i(y) - b_i(z)] [K(y, z) - K(y, x_R)] \right. \right. \\
& \quad \left. \left. \times h(z) d\mu(z) \right|^p d\mu(y) \right\}^{1/p} \\
& \leq C \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=2}^{\infty} \mu(2^{k+1}R)^{1/p'} \left\{ \int_{2^{k+2}R \setminus 2^{k-1}R} |[b(y) - m_{\tilde{R}}(b)]_{\sigma}|^p \right. \\
& \quad \left. \times \left| \int_R [m_{\tilde{R}}(b) - b(z)]_{\sigma'} [K(y, z) - K(y, x_R)] h(z) d\mu(z) \right|^p d\mu(y) \right\}^{1/p} \\
& \leq C \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=2}^{\infty} \frac{\mu(2^{k+1}R)^{1/p'} l(R)^{\delta}}{l(2^k R)^{n+\delta}} \left\{ \int_{2^{k+2}R} |[b(y) - m_{\tilde{R}}(b)]_{\sigma}|^p d\mu(y) \right\}^{1/p} \\
& \quad \times \int_R |[m_{\tilde{R}}(b) - b(z)]_{\sigma'}| |h(z)| d\mu(z) \\
& \leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \|\vec{b}_{\sigma}\|_* \sum_{k=2}^{\infty} \frac{\mu(2^{k+3}R) l(R)^{\delta}}{l(2^k R)^{n+\delta}} [K_{\widetilde{R}, 2^{k+2}R}]^i \\
& \quad \times \left\{ \int_{Q_j} |a_j(z)|^p d\mu(z) \right\}^{1/p} \left\{ \int_{Q_j} |[m_{\tilde{R}}(b) - b(z)]_{\sigma'}|^{p'} d\mu(z) \right\}^{1/p'} \\
& \leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} [K_{Q_j, R}]^m \\
& \leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j|.
\end{aligned}$$

An argument similar to the estimate for Π_{21} can also give us a desired estimate for Π_{22} .

Finally, we estimate Π_{23} . By (b) and (c) of Definition 1.2, (ii) of Definition 1.3, (2.2), (1.2), the Hölder inequality, Lemma 2.2 and (3.1), we obtain

$$\begin{aligned}
 \Pi_{23} &\leq \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^k R} \sum_{l=k+2}^{\infty} \int_{2^{l+1}R \setminus 2^l R} \int_R |K(y, z) - K(y, x_R)| \prod_{i=1}^m |b_i(y) - b_i(z)| \\
 &\quad \times |h(z)| d\mu(z) \left\{ \frac{1}{|y - x|^n} + \frac{1}{|x_R - x|^n} \right\} d\mu(y) d\mu(x) \\
 &\leq C \sum_{j=1}^2 |\lambda_j| \sum_{i=0}^m \sum_{\sigma \in C_i^m} \sum_{k=2}^{\infty} \sum_{l=k+2}^{\infty} \frac{\mu(2^{k+1}R)}{l(2^k R)^n} \frac{l(R)^\delta}{l(2^l R)^{n+\delta}} \int_{2^{l+1}R} |[b(y) - m_{\tilde{R}}(b)]_\sigma| d\mu(y) \\
 &\quad \times \int_{Q_j} |[m_{\tilde{R}}(b) - b(z)]_{\sigma'}| |a_j(z)| d\mu(z) \\
 &\leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^p(\mu)} \mu(2Q_j)^{1/p'} \sum_{i=0}^m \sum_{\sigma \in C_i^m} [K_{Q_j R}]^{m-i} \\
 &\quad \times \sum_{k=2}^{\infty} \sum_{l=k+2}^{\infty} \frac{\mu(2^{k+1}R)}{l(2^k R)^n} \frac{\mu(2^{l+2}R)l(R)^\delta}{l(2^l R)^{n+\delta}} [K_{\tilde{R}, 2^{l+1}R}]^i \\
 &\leq C \|\vec{b}\|_* \sum_{j=1}^2 |\lambda_j|.
 \end{aligned}$$

An argument similar to the estimate for Π_{23} can also give us a desired estimate for Π_{24} .

Combining the estimates for Π_{21} , Π_{22} , Π_{23} and Π_{24} , we obtain a desired estimate for Π_2 . The estimates for Π_1 and Π_2 tell us that

$$\Pi = \int_{\mathbb{R}^d \setminus 4R} M_\Phi[T_{\vec{b}}(h)](x) d\mu(x) \leq C |b|_{H_{\vec{b}, m, m+2}^{1,p}(\mu)}. \quad (3.5)$$

The estimates (3.3) and (3.5) lead to (3.2) and this completes the proof of our theorem. \square

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